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A Geometric Theory of Plasticity

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Summary: *A new geometric formulation of rate-independent generalized plasticity is presented. The formulation relies crucially on the consideration of the physical (referential) metric as a primary internal variable and does not invoke any decomposition of the kinematical quantities into elastic and plastic parts. On the basis of a purely geometrical argument the transition to classical plasticity is demonstrated. The covariant balance of energy is systematically employed for the derivation of the mechanical state equations. It is shown that unlike the case of finite elasticity, in finite plasticity, the covariant balance of energy does not yield the Doyle-Ericksen formula, unless a further assumption is made. As an application, a new material model is developed and is tested numerically for the solution of several problems of large scale plastic flow.*

1 Introduction

The kinematics of large deformation plasticity has been for a long time a subject of serious debate. For instance, Nemat-Nasser [1] on the basis of the principle of energy conservation proposed an additive decomposition of the rate of deformation tensor. This decomposition constitutes the basic kinematical assumption adopted in several works of computational interest, such as those of Nagtegaal and De Jong [2], Key and Krieg [3], Atluri [4] and Panoskaltsis et al. [5]. Green and Naghdi [6, 7] considered the additive decomposition of the Lagrangian strain tensor, while Lee and Liu [8] and Lee [9] among others (e.g. Mandel [10], Kratochvil [11], Dashner [12], Lubliner [13, 14], Simo [15], Le and Stumpf [16]) advocated the multiplicative decomposition of the deformation gradient together with the concept of an elastically relaxed (intermediate) configuration. Additional suggestions constitute the combination of both additive and multiplicative theories proposed by Simo and Ortiz [17]. However, as it is pointed out in a review paper by Naghdi [19], almost all authors introduce some measure of plastic strain that can be either a primitive quantity or a solution of an evolution equation, from which the plastic strain is obtained after suitable initial conditions have been specified. Such an assumption has been subjected to the serious questioning of Gilman [20], among others. The basic objective of this paper is to propose a new formulation of the theory of plasticity, in the general case of finite deformations, which is not based on the introduction of an "artificial" quantity that will stand for plastic strain, but on the *referential metric* which is a natural quantity. The powerful concept of referential metric was introduced by Valanis [22] (who termed it "physical metric") and it was advanced further by Valanis and Panoskaltsis [23]. The approach has its origins in the realization that since the body is embedded into some ambient space, it is automatically endowed with two distinct metrics, namely the ambient space metric and the body one. These metrics are of different thermomechanical origin and their interrelation in the course of deformation will inevitably specify both the elastic and the plastic (dissipative) properties of the body. As a result, the body (referential) metric is considered as the primary internal variable, while the metric of the ambient space is considered as the control variable, leading naturally to a strain space approach, which offers several advantages in the formulation of an elastic-plastic theory (see, for instance Naghdi [19]). Moreover, in our work we introduce non-Euclidean spaces providing some important advantages, that are not supplied by the classical Euclidean ones. In particular, the present approach can describe several internal material structures, which may differ vastly from the classical Euclidean ones, namely directional densities, curved material structures, pre-formed materials, pre- stressed reference configurations and the presence of dislocation fields, which may change a Euclidean internal structure to a non-Euclidean one (see, Valanis and Panoskaltsis [23]). Second, the introduction of manifold underlying spaces, besides allowing for the extraction of valuable geometrical information, provides the necessary mathematical tools for a covariant formulation of the theory. Generalized plasticity theory (Lubliner [13, 14, 21]) is used in this paper, since it is more general and versatile than classical plasticity. Also, as we have proved in another publication [41], classical plasticity is a particular case of generalized plasticity theory.

The present paper is organized as follows: In section 2 generalized plasticity is presented in a covariant setting. For this purpose, manifold structure is considered not only for the body of interest and the ambient space, but also for the state space, that is the set of all realizable states over a material point. The involvement of the standard pull-back/push-forward operations (e.g. Marsden and Hughes [24, p. 67]), leads to the introduction of the convected Lie derivative (e.g. [24, p. 95], which eventually leads to a covariant formulation of the theory. Loading-unloading criteria, in both the reference and the spatial configurations, are derived as well. The spatial covariant balance of energy, which constitutes the keystone for any covariant theory, relativistic or non–relativistic, is presented in section 3. It is emphasized that even though this concept has been extensively studied within the context of finite elasticity this is not the case within the context of an inelastic theory. Our development shows that, unlike the case of finite elasticity, the covariant balance of energy does not yield the Doyle-Ericksen formula, unless a further assumption is made. As an application, a material model is derived and its predictions for several problems of large scale plastic flow are presented in section 4.

2 Constitutive theory

Since we deal with large scale plastic flow, we follow the geometrical approach proposed within the context of non-linear elasticity by Marsden and Hughes [24]. Accordingly, we consider both the body of interest and the ambient space S as three dimensional Riemannian manifolds with (covariant) metrics **G** and **g**, respectively. In particular, let Ω be the reference configuration of the body of interest with points labeled by $\left(X^1, X^2, X^3\right)$ and define a motion of the body within the ambient space as the time dependent mapping **x**:

$$\begin{aligned} \mathbf{x}:\Omega \to S, \quad x^1 &= x^1\left(X^1, X^2, X^3, t\right), \\ x^2 &= x^2\left(X^1, X^2, X^3, t\right), \\ x^3 &= x^3\left(X^1, X^2, X^3, t\right), \end{aligned} \tag{1}$$

which maps the points of the reference configuration onto the points $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ of the current (spatial) configuration.

Then the deformation gradient is defined as the tangent map of (1), i.e.

$$\mathbf{F}\left(\mathbf{X}, t\right) = \frac{\partial \mathbf{x}\left(\mathbf{X}, t\right)}{\partial \mathbf{X}}, \tag{2}$$

with determinant $J\left(\mathbf{X}, t\right) = \det\left[\mathbf{F}\left(\mathbf{X}, t\right)\right] > 0.$ Additionally, we consider the right Cauchy-Green deformation tensor defined as the pull back of the spatial metric **g** and the Finger deformation tensor **c** defined as the push-forward of the referential metric **G**:

$$\mathbf{C} = \mathbf{x}^*(\mathbf{g}) = \mathbf{F}^{\mathrm{T}}\mathbf{g}\,\mathbf{F}, \tag{3}$$

$$\mathbf{c} = \mathbf{x}_*\left(\mathbf{G}\right) = \mathbf{F}^{\text{-}\mathrm{T}}\mathbf{G}\,\mathbf{F}^{\mathrm{T}}. \tag{4}$$

A detailed analysis of the geometrical meaning of those quantities can be found in Valanis [22] (see also Yavari et al. [28]).

Generalized plasticity (e.g. Lubliner [13, 14, 21]) is a *local* internal variable theory of rate independent behavior, which is based primarily on the assumption that plastic deformation may take place on loading but not on unloading. The theory may, in principle, be formulated equivalently with respect to the stress or the strain (deformation) space. Since we deal with manifold underlying spaces and their corresponding metrics, a strain space formulation of the theory seems more natural. In turn, in the absence of thermal effects[1] the mechanical state at the referential point **X** with coordinates X^1, X^2, X^3 is determined by the control variable, which is identified by the right Cauchy-Green tensor and the internal variable vector. The latter is assumed to be composed by the referential metric **G** and an additional internal variable vector **Q**. In general, the referential metric is unknown and several considerations must be made for its determination, including experimental procedures (Valanis and Panoskaltsis [23]). To this end, it is emphasized that for the elastic-plastic (dissipative) continuum studied here, the referential metric is a function of the history of deformation (see Valanis [22], Valanis and Panoskaltsis

[1] The extension of the theory to the non-isothermal regime is a non-trivial task and may be done along the lines presented in this work in conjunction with some developments given in Marsden and Hughes [24] and Lubliner [14].

[23]) and its consideration among the internal variables has a concrete physical basis. It is also emphasized that the only one case in which the referential metric is constant in the course of deformation is that of an elastic (non-dissipative) material, like the one discussed in the covariant approach of Marsden and Hughes [24]. The additional internal variable vector may be composed by hardening parameters (e.g. [15, 30]), evolving anisotropy directions (e.g. [29]) and multiple material metrics arising as a physical possibility when changes in the internal structure are due to multiple internal deformation mechanisms [22]. The internal variable vector **Q** is assumed to be covariant in the sense that under mapping (1) obeys the general transformation law, that is:

$$q^{j_1 \dots j_r}{}_{i_1 \dots i_s} = \frac{\partial x^{j_1}}{\partial X^{I_1}} \dots \frac{\partial x^{j_r}}{\partial X^{I_r}} \frac{\partial X^{K_1}}{\partial x^{i_1}} \dots \frac{\partial X^{K_s}}{\partial x^{i_s}} Q^{I_1 \dots I_r}{}_{K_1 \dots K_s}, \tag{5}$$

where $q^{j_1 \dots j_r}{}_{i_1 \dots i_s}$ are the components of **q** which is the push-forward of **Q,** i.e. $\mathbf{q} = \mathbf{x}_*(\mathbf{Q})$.

The state space $\boldsymbol{\mathscr{D}}$**=(C, G, Q)** is assumed to be attached at the point **X** so that the set $\{\mathbf{X}\} \times \boldsymbol{\mathscr{D}}$ is a fiber of **X**. A dynamical process **P** may be identified by the local vector bundle mapping (e.g. Abraham et al. [27, p. 167]):

$$\mathbf{P} : \Omega \times \boldsymbol{\mathscr{D}} \to S \times \boldsymbol{\mathscr{D}}',$$

defined as:

$$\mathbf{P}\Big((\mathbf{X},t),\mathbf{C},\mathbf{G},\mathbf{Q}\Big) = \Big((\mathbf{x},t),\mathbf{x}_*(\mathbf{C}),\mathbf{x}_*(\mathbf{G}),\mathbf{x}_*(\mathbf{Q})\Big) = \Big((\mathbf{x},t),\mathbf{g},\mathbf{c},\mathbf{q}\Big). \tag{6}$$

Accordingly, the material state in the current configuration at the point **x** with coordinates x^1, x^2, x^3 is determined by the spatial metric **g** and the internal variable vector (**c**, **q**). A *local process* $\boldsymbol{\Psi}$ in the state space $\boldsymbol{\mathscr{D}}$ is defined as a curve in $\boldsymbol{\mathscr{D}}$, i.e. as a mapping:

$$\boldsymbol{\Psi} : I \in \mathrm{R} \to \boldsymbol{\mathscr{D}}$$

defined as:

$$\boldsymbol{\Psi}(t) = \Big(\mathbf{C}(t),\mathbf{G}(t),\mathbf{Q}(t)\Big).$$

The direction and the speed of such a process is determined by the tangent vector,

$$\dot{\boldsymbol{\Psi}} : \boldsymbol{\mathscr{D}} \to T\boldsymbol{\mathscr{D}}, \text{ with } \dot{\boldsymbol{\Psi}}(t) = (\dot{\mathbf{C}}(t),\dot{\mathbf{G}}(t),\dot{\mathbf{Q}}(t)),$$

or simply $\dot{\boldsymbol{\Psi}} = (\dot{\mathbf{C}},\dot{\mathbf{G}},\dot{\mathbf{Q}})$, where $T\boldsymbol{\mathscr{D}}$ is the tangent space of $\boldsymbol{\mathscr{D}}$. Since $\dot{\mathbf{C}}$ is always known under deformation control, the components $(\dot{\mathbf{G}},\dot{\mathbf{Q}})$ have to be determined. The latter are assumed to be given as:

$$\begin{aligned} \dot{\mathbf{G}} &= \mathbf{A}\Big(\mathbf{C},\mathbf{G},\mathbf{Q},\dot{\mathbf{C}}\Big), \\ \dot{\mathbf{Q}} &= \mathbf{B}\Big(\mathbf{C},\mathbf{G},\mathbf{Q},\dot{\mathbf{C}}\Big). \end{aligned} \tag{7}$$

or equivalently,

$$\begin{aligned} L_v\mathbf{c} &= \mathbf{a}\Big(\mathbf{g},\mathbf{c},\mathbf{q},\mathbf{F},L_v\mathbf{g}\Big), \\ L_v\mathbf{q} &= \mathbf{b}\Big(\mathbf{g},\mathbf{c},\mathbf{q},\mathbf{F},L_v\mathbf{g}\Big), \end{aligned} \tag{8}$$

where $\mathbf{A}$ and $\mathbf{B}$: $T\boldsymbol{\mathscr{D}} \times \boldsymbol{\mathscr{D}} \to T\boldsymbol{\mathscr{D}}$ and $\mathbf{a}$ and $\mathbf{b} : T\boldsymbol{\mathscr{D}}' \times \boldsymbol{\mathscr{D}}' \to T\boldsymbol{\mathscr{D}}'$ are vector fields in $T\boldsymbol{\mathscr{D}}$ and $T\boldsymbol{\mathscr{D}}'$, respectively and hence they are considered as tensorial functions of the denoted arguments and $L_v(\cdot)$ stands for the Lie derivative, defined as the convected derivative relative to the current configuration (e.g., [25, 27]). Further, the dependence of the functions $\mathbf{a}$, $\mathbf{b}$ on the deformation gradient **F,** because of the push-forward operation by which Equation (8) is derived from Equation (7) is noteworthy.

Rate independence implies that Equations (7) and (8) are invariant under a replacement of t by $\chi(t)$ where $\chi(\cdot)$ is any monotonically increasing function. Then a necessary and sufficient condition for rate independence is that **A** and **B** be homogeneous to the first degree (see [14]), that is for any positive number α :

$$\begin{aligned}\mathbf{A}(\mathbf{C},\mathbf{G},\mathbf{Q},\alpha\cdot\dot{\mathbf{C}}) &= \alpha\cdot\mathbf{A}(\mathbf{C},\mathbf{G},\mathbf{Q},\dot{\mathbf{C}}),\\ \mathbf{B}(\mathbf{C},\mathbf{G},\mathbf{Q},\alpha\cdot\dot{\mathbf{C}}) &= \alpha\cdot\mathbf{B}(\mathbf{C},\mathbf{G},\mathbf{Q},\dot{\mathbf{C}})\end{aligned} \tag{9}$$

or equivalently in the spatial description:

$$\begin{aligned}\mathbf{a}(\mathbf{g},\mathbf{c},\mathbf{q},\alpha\cdot L_v\mathbf{g}) &= \alpha\cdot\mathbf{a}(\mathbf{g},\mathbf{c},\mathbf{q},\mathbf{F},L_v\mathbf{g}),\\ \mathbf{b}(\mathbf{g},\mathbf{c},\mathbf{q},\alpha\cdot L_v\mathbf{g}) &= \alpha\cdot\mathbf{b}(\mathbf{g},\mathbf{c},\mathbf{q},\mathbf{F},L_v\mathbf{g}).\end{aligned} \tag{10}$$

A local process is defined as elastic if it lies entirely in a six dimensional submanifold of $\mathfrak{D}$ defined by **(G**, **Q) =** constant; otherwise is defined as plastic. The basic concept of generalized plasticity is that of the elastic range of a state (e.g. Pipkin and Rivlin [31], Lucchesi and Podio-Guidugli [32], Bertram [33]). The latter is introduced as a submanifold of the state space, defined as the manifold comprising the values of **C** that can be reached by an elastic process from the current state. It is further assumed that the boundary of the elastic range is a five dimensional manifold, the points of which have coordinate neighborhoods, and which is attached to the interior in much the same way a face of a cube is attached to the interior. The latter manifold may be defined as a loading surface at (**G**, **Q**) (e.g., Lubliner [14], Eisenberg and Phillips [34]). A state within the elastic range may be defined as plastic if it lies on a loading surface and elastic otherwise. Accordingly, the rate equations for the internal variables may be derived on the basis of the defining property of a plastic state and the irreversibility of a process from such a state. Thus, if (**C**, **G**, **Q**) is a plastic state and **N** is the outward normal to the loading surface in the state under consideration, then a simple form for the functions **A** and **B** which fulfills both requirements together with the homogeneity conditions is:

$$\begin{aligned}\mathbf{A} &= \Gamma\mathbf{\Lambda}(\mathbf{C},\mathbf{G},\mathbf{Q})\left\langle \mathbf{N}:\dot{\mathbf{C}}\right\rangle,\\ \mathbf{B} &= \Gamma\mathbf{M}(\mathbf{C},\mathbf{G},\mathbf{Q})\left\langle \mathbf{N}:\dot{\mathbf{C}}\right\rangle,\end{aligned} \tag{11}$$

where Γ is a scalar function of the state variables related to the yielding properties of the continuum and which must be positive at any plastic state and zero at any elastic one and <·> stands for the Macauley bracket defined as:

$$\left\langle x\right\rangle = \begin{cases} x & \text{if } x > 0\\ 0 & \text{if } x \le 0.\end{cases}$$

Finally, **Λ** and **M** are assumed to be non-vanishing functions of the state variables which are associated with the direction of the plastic flow.

In view of Equations (11) the final form for the rate equations for the internal variables may be stated as:

$$\begin{aligned}\dot{\mathbf{G}} &= \Gamma\mathbf{\Lambda}(\mathbf{C},\mathbf{G},\mathbf{Q})\left\langle \mathbf{N}:\dot{\mathbf{C}}\right\rangle,\\ \dot{\mathbf{Q}} &= \Gamma\mathbf{M}(\mathbf{C},\mathbf{G},\mathbf{Q})\left\langle \mathbf{N}:\dot{\mathbf{C}}\right\rangle.\end{aligned} \tag{12}$$

From the rate Equations (12) one can derive directly the loading-unloading criteria for generalized plasticity theory as:

$$\begin{cases} \Gamma(\mathbf{C},\mathbf{G},\mathbf{Q}) = 0 & \text{elastic state,} \\ \Gamma(\mathbf{C},\mathbf{G},\mathbf{Q}) \neq 0 & \text{and } \mathbf{N}{:}\dot{\mathbf{C}} < 0 \text{ elastic process,} \\ & \phantom{\text{and }} \mathbf{N}{:}\dot{\mathbf{C}} = 0 \text{ neutral process,} \\ & \phantom{\text{and }} \mathbf{N}{:}\dot{\mathbf{C}} > 0 \text{ plastic process.} \end{cases} \tag{13}$$

To this end it is emphasized that the loading-unloading criteria play a paramount role for the numerical implementation for a generalized plasticity based model (see Panoskaltsis et al. [5, 30]).

The equivalent spatial formulation can be derived either in a similar manner (see [5]), or by performing a push-forward operation to Equations (12) as:

$$\begin{aligned} L_v\mathbf{c} &= \gamma\boldsymbol{\lambda}(\mathbf{g},\mathbf{c},\mathbf{q},\mathbf{F})\left\langle \mathbf{n}{:}L_v\mathbf{g}\right\rangle, \\ L_v\mathbf{q} &= \gamma\boldsymbol{\mu}(\mathbf{g},\mathbf{c},\mathbf{q},\mathbf{F})\left\langle \mathbf{n}{:}L_v\mathbf{g}\right\rangle, \end{aligned} \tag{14}$$

where $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and $\mathbf{n}$ are the push-forwards of the functions $\boldsymbol{\Lambda}$, $\mathbf{M}$, $\mathbf{N}$ in the current configuration and γ is the expression of Γ in terms of the spatial variables. The loading - unloading criteria follow in a similar manner as:

$$\begin{cases} \gamma(\mathbf{g},\mathbf{c},\mathbf{q},\mathbf{F}) = 0 & \text{elastic state,} \\ \gamma(\mathbf{g},\mathbf{c},\mathbf{q},\mathbf{F}) \neq 0 & \text{and } \mathbf{n} : L_v\mathbf{g} < 0 \text{ elastic process,} \\ & \phantom{\text{and }} \mathbf{n} : L_v\mathbf{g} = 0 \text{ neutral process,} \\ & \phantom{\text{and }} \mathbf{n} : L_v\mathbf{g} > 0 \text{ plastic process.} \end{cases} \tag{15}$$

The rate Equations (12) or (14) along with the mathematical expression of the loading (hyper) surfaces in the state space, which in general are assumed to be given as single parameter families of the form:

$$\Phi(\mathbf{C},\mathbf{G},\mathbf{Q}) = K, \tag{16}$$

or equivalently:

$$\varphi(\mathbf{g},\mathbf{c},\mathbf{q},\mathbf{F}) = k, \tag{17}$$

constitute the basic ingredients of rate-independent generalized plasticity. It is concluded that the theory does not employ the concept of the yield surface as a basic ingredient. Nevertheless, the concept of the yield surface may be introduced on the basis of purely geometric arguments and an equivalent approach for classical plasticity may be obtained.

3 The covariant balance of energy and the stress tensor

The keystone element of any covariant constitutive theory, relativistic or non- relativistic, is that of the *covariant balance of energy*. A first approach to this concept, within the context of non-linear elasticity and manifold underlying spaces, is given in Marsden and Hughes [24] (see also Simo and Marsden [26]). In particular, these authors by *postulating the invariance of the energy density under arbitrary spatial diffeomorphisms,* that is spatial transformations which may change the whole ambient space, derived the conservation and balance laws of classical mechanics, together with the Doyle-Ericksen formula (see Doyle and Ericksen [36]). The concept has been studied further by Yavari et al. [28], where particular emphasis is placed on the transformation properties of the balance of energy under *arbitrary referential diffeomorphisms;* this approach leads to the notion of the *configurational forces,* which are forces acting on the reference configuration. Nevertheless, the concept seems to have passed largely unnoticed within the context of materials with internal variables, which do not appear explicitly in the balance laws (Bertram [33]). Its generalization for the elastic-plastic body studied herein, constitutes the primary objective of this section.

The principle of balance of energy may be stated as an axiom as follows:

Let U be any open subset of Ω and let $\mathbf{P}: U \times \mathscr{D} \to S \times \mathscr{D}'$ be a fixed dynamical process of the body within the space $S \times \mathscr{D}'$, which is modeled by the local vector bundle map (6). Let $\rho(\mathbf{x}, t)$ be the mass density, $e(\mathbf{x}, t)$ the internal energy function per unit mass, $\mathbf{v}(\mathbf{x}, t)$ the spatial velocity, $\mathbf{b}(\mathbf{x}, t)$ the external body force per unit mass and $\mathbf{t}(\mathbf{x}, t, \mathbf{m})$ the Cauchy traction vector, where $\mathbf{m}$ is the unit normal to the boundary $\partial\mathbf{x}(U)$. Then, for a purely mechanical theory the balance of energy axiom reads as follows:

Axiom (Balance of Energy): The dynamical process satisfies balance of energy if:

$$\frac{d}{dt}\int_{x(U)} \rho\left(e + \tfrac{1}{2}\left\langle \mathbf{v}, \mathbf{v}\right\rangle\right) dv = \int_{x(U)} \rho\left(\left\langle \mathbf{b}, \mathbf{v}\right\rangle\right) dv + \int_{\partial x(U)} \left\langle \mathbf{t}, \mathbf{m}\right\rangle da \tag{24}$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product in S and dv, da are the volume and area elements respectively, in the current configuration. The covariant balance of energy axiom *within the context of the elastic-plastic body* in question may be stated as follows:

Axiom (Covariant Balance of Energy): For the fixed dynamical process $\mathbf{P}: U \times \mathscr{D} \to S \times \mathscr{D}'$ which satisfies the balance of energy axiom, consider an arbitrary superposed spatial diffeomorphism $\boldsymbol{\xi}: (S, \mathscr{D}') \to (S, \mathscr{D}')$. Postulate that the new dynamical process $\overline{\mathbf{x}} = \boldsymbol{\xi} \circ \mathbf{x}$ satisfies the balance of energy axiom, *provided that the metric $\boldsymbol{g}$ is replaced by $\boldsymbol{\xi}^*\mathbf{g}$, the Finger deformation tensor $\boldsymbol{c}$ is replaced by $\boldsymbol{\xi}^*\mathbf{c}$, the additional internal variable vector $\boldsymbol{q}$ is replaced by* $\boldsymbol{\xi}^*\mathbf{q}$ and velocities, forces, accelerations, etc. are transformed according to the standard laws of tensorial calculus (e.g., Marsden and Hughes [24, p. 163]), i.e.:

$$\frac{d}{dt}\int_{\overline{x}(U)} \overline{\rho}\left(\overline{e} + \tfrac{1}{2}\left\langle \overline{\mathbf{v}}, \overline{\mathbf{v}}\right\rangle\right) d\overline{v} = \int_{\overline{x}(U)} \overline{\rho}\left(\left\langle \overline{\mathbf{b}}, \overline{\mathbf{v}}\right\rangle\right) d\overline{v} + \int_{\partial \overline{x}(U)} \left\langle \overline{\mathbf{t}}, \overline{\mathbf{m}}\right\rangle d\overline{a}. \tag{25}$$

We assume that unlike the classical elastic case discussed for instance in Marsden and Hughes [24], where the functions e, ρ, etc. depend solely on the spatial metric, in the case of an inelastic material with internal variables, these functions depend on the internal variables as well. This plausible consideration follows from the fact that the internal variables -even though not controllable- they are measurable (e.g. Maugin [37, p. 277]) and play an equally weighted role to the continuum response.

The basic assumption for postulating Equation (25) lies crucially on the adopted transformation law for the internal energy density, namely:

$$\bar{\mathbf{x}} = \boldsymbol{\xi}(\mathbf{x}), \quad \bar{e}(\bar{\mathbf{x}}, t, \mathbf{g}, \mathbf{c}, \mathbf{q}) = e(\mathbf{x}, t, \boldsymbol{\xi}^{*}\mathbf{g}, \boldsymbol{\xi}^{*}\mathbf{c}, \boldsymbol{\xi}^{*}\mathbf{q})\,. \tag{26}$$

As it is noted in Simo and Marsden [26], this assumption is rather natural and is based on the fact that referential tensors such as the right Cauchy-Green tensor **C,** the referential metric **G** and the additional internal variable vector **Q**, remain unchanged under the superposed spatial diffeomorphism. In particular, if $\boldsymbol{\xi} : (S, \mathbf{g}, \mathbf{c}, \mathbf{q}) \to (S, \bar{\mathbf{g}}, \bar{\mathbf{c}}, \bar{\mathbf{q}})$ is a spatial diffeomorphism, the new metric $\bar{\mathbf{g}}$ and the new internal variable vector $(\bar{\mathbf{c}}, \bar{\mathbf{q}})$ must be such that $\bar{\mathbf{C}} = \mathbf{C}$ and $(\bar{\mathbf{G}}, \bar{\mathbf{Q}}) = (\mathbf{G}, \mathbf{Q})$, that is:

$$\bar{\mathbf{g}} = \boldsymbol{\xi} \circ \mathbf{x}_{*}(\bar{\mathbf{C}}) = \boldsymbol{\xi} \circ \mathbf{x}_{*}(\mathbf{C}) = \boldsymbol{\xi}_{*}(\mathbf{g}) \tag{27}$$

and similarly,

$$(\bar{\mathbf{c}}, \bar{\mathbf{q}}) = \boldsymbol{\xi} \circ \mathbf{x}_{*}[(\bar{\mathbf{G}}, \bar{\mathbf{Q}})] = \boldsymbol{\xi} \circ \mathbf{x}_{*}[(\mathbf{G}, \mathbf{Q})] = \boldsymbol{\xi}_{*}(\mathbf{c}, \mathbf{q}). \tag{28}$$

Furthermore, by extending the classical elastic case, one can define the set

$$O_{(\mathbf{g},\mathbf{c},\mathbf{q})} = \left\{ \left(\boldsymbol{\xi}_{*}(\mathbf{g}), \boldsymbol{\xi}_{*}(\mathbf{c}, \mathbf{q}) \right) \mid \boldsymbol{\xi} : (S, \boldsymbol{\mathcal{D}}') \to (S, \boldsymbol{\mathcal{D}}') \text{ is a diffeomorphism} \right\}$$

as the orbit of the state variables.

The covariance axiom may be systematically used in order to find restrictions on constitutive equations of materials and in particular, when deformation is the primary variable, to show how the stress can be derived from the internal energy density. For instance, Doyle and Ericksen [36, p. 57] proved that for an elastic continuum, the Cauchy stress tensor $\boldsymbol{\sigma}$ is given as:

$$\boldsymbol{\sigma} = 2\rho \frac{\partial e}{\partial \mathbf{g}}\,. \tag{29}$$

For the rate independent continuum, with the rate equations taking the form (14), the procedure is the following: Equation (25) is evaluated at time $t = t_0$ for which

$$\boldsymbol{\xi}\Big|_{t=t_0} = \text{identity and } \mathbf{w} = \frac{\partial \boldsymbol{\xi}}{\partial t}\Big|_{t=t_0},$$

is the velocity of $\boldsymbol{\xi}$ (at $t = t_0$).

Then, by applying a standard procedure (see Marsden and Hughes [24, pp. 166, 167]), which involves the transport theorem, the divergence theorem and the Cauchy tetrahedron, we obtain the equations of conservation of mass, balance of momentum, balance of moment of momentum, together with the additional identity:

$$\int_{x(U)} \left[\rho \left(\dot{\bar{e}} - e \right) - \tfrac{1}{2} \boldsymbol{\sigma} : L_{\mathbf{w}} \mathbf{g} \right] dv = 0. \tag{30}$$

The definition of the Lie derivative yields:

$$\begin{aligned} \dot{\bar{e}} &= \dot{e} + \frac{\partial e}{\partial \mathbf{g}} : \frac{d}{dt}\Big|_{t=t_0} \boldsymbol{\xi}^{*}(\mathbf{g}) + \frac{\partial e}{\partial \mathbf{c}} : \frac{d}{dt}\Big|_{t=t_0} \boldsymbol{\xi}^{*}(\mathbf{c}) + \frac{\partial e}{\partial \mathbf{q}} : \frac{d}{dt}\Big|_{t=t_0} \boldsymbol{\xi}^{*}(\mathbf{q}) = \\ &= \dot{e} + \frac{\partial e}{\partial \mathbf{g}} : L_{\mathbf{w}} \mathbf{g} + \frac{\partial e}{\partial \mathbf{c}} : L_{\mathbf{w}} \mathbf{c} + \frac{\partial e}{\partial \mathbf{q}} : L_{\mathbf{w}} \mathbf{q}. \end{aligned} \tag{31}$$

Substitution of (31) into (30) yields:

$$\int_{x(U)} \left[\rho \frac{\partial e}{\partial \mathbf{g}} : L_{\mathrm{w}} \mathbf{g} + \rho \frac{\partial e}{\partial \mathbf{c}} : L_{\mathrm{w}} \mathbf{c} + \rho \frac{\partial e}{\partial \mathbf{q}} : L_{\mathrm{w}} \mathbf{q} - \tfrac{1}{2} \boldsymbol{\sigma} : L_{\mathrm{w}} \mathbf{g} \right] dv = 0. \tag{32}$$

Substitution of the rate Equations (12) into Equation (31) yields

$$\int_{x(U)} \left\{ \left[\rho \frac{\partial e}{\partial \mathbf{g}} + \rho \frac{\partial e}{\partial \mathbf{c}} : \boldsymbol{\lambda}(\mathbf{g}, \mathbf{c}, \mathbf{q}, \mathbf{F}) \mathbf{n} + \rho \frac{\partial e}{\partial \mathbf{q}} : \boldsymbol{\mu}(\mathbf{g}, \mathbf{c}, \mathbf{q}, \mathbf{F}) \mathbf{n} - \tfrac{1}{2} \boldsymbol{\sigma} \right] : L_{\mathrm{w}} \mathbf{g} \right\} dv = 0\,, \tag{33}$$

from which and by noting that $L_{\mathrm{w}}\mathbf{g}$ can be arbitrarily specified, we derive

$$\rho \frac{\partial e}{\partial \mathbf{g}} + \rho \frac{\partial e}{\partial \mathbf{c}} : \boldsymbol{\lambda}(\mathbf{g}, \mathbf{c}, \mathbf{q}, \mathbf{F}) \mathbf{n} + \rho \frac{\partial e}{\partial \mathbf{q}} : \boldsymbol{\mu}(\mathbf{g}, \mathbf{c}, \mathbf{q}, \mathbf{F}) \mathbf{n} - \frac{1}{2} \boldsymbol{\sigma} = 0\,. \tag{34}$$

Therefore, the covariance axiom does not yield the Doyle-Ericksen formula, unless a further assumption is made, namely that an unloading process (i.e. a process with $\mathbf{n}{:}L_v \mathbf{g} \le 0$) from a plastic state is *quasi-reversible,* which means that in such a process the plastic dissipation, defined as

$$d_p = -\frac{\partial e}{\partial \mathbf{c}} : L_v \mathbf{c} - \frac{\partial e}{\partial \mathbf{q}} : L_v \mathbf{q} \left(= -\frac{\partial e}{\partial \mathbf{G}} : \dot{\mathbf{G}} - \frac{\partial e}{\partial \mathbf{Q}} : \dot{\mathbf{Q}} \right)$$

vanishes. If this is the case, the Doyle-Ericksen formula can be derived as in the classical elastic case directly from Equation (32).

It is interesting to note that Lubliner [14] arrives at a similar result by working entirely in the reference configuration and on the basis of the second law of thermodynamics, expressed in the form of the Clausius-Plank inequality. In our covariant approach we bypass the second law, focusing on all transformations of a given process, unlike the second law of thermodynamics where we focus on all processes (Marsden and Hughes [24, p. 201]).

4 A model problem

In order to illustrate the application of the presented concepts to the constitutive modeling of solid materials a specific model is developed. The formulation of the model is motivated by classical metal plasticity and in particular it comprises von-Mises loading surfaces with both isotropic and kinematical hardening.

Since we deal with large scale inelastic flow, the kinematics of the problem together with the covariance principle, suggest that a formulation of the model in terms of the spatial metrics and their Lie derivatives is more fundamental. Further, in the current configuration the spatial metric has usually a diagonal form, which makes the computations simpler than those in the reference configuration, where the (Lagrangian) metric **C** is fully populated (e.g. see [29]). Accordingly, the stress response is assumed to be hyperelastic, governed by an isotropic strain energy function, which is given in terms of the invariants of the tensor $\mathbf{gb}$, where **b** is the left Cauchy-Green tensor, defined as the push-forward of the reciprocal (contravariant) metric $\mathbf{G}^{-1}$, i.e. $\mathbf{b} = \mathbf{x}_* \left(\mathbf{G}^{-1} \right) = \mathbf{F} \mathbf{G}\, \mathbf{F}^{\mathrm{T}}$, as:

$$\rho_0 \psi = \lambda \frac{(I_3)^2 - 1}{4} - \frac{\lambda}{2} \ln \sqrt{I_3} + \frac{1}{2} \mu I_3^{-\frac{1}{3}} \left[I_1 - 3 \right],$$

where λ and μ are Lame' type of parameters, ρ_0 is the referential density and I_1 and I_3 are the first and the third invariants of the tensor $\mathbf{gb}$.
Then, the Kirchhoff stress tensor $\boldsymbol{\tau}$, $(\boldsymbol{\tau} = J\boldsymbol{\sigma})$ is given by the Doyle-Ericksen formula (see [24, p. 204]) as:

$$\boldsymbol{\tau} = 2\rho_0 \frac{\partial \psi}{\partial \mathbf{g}} = \lambda \frac{I_3 - 1}{2} \mathbf{g}^{-1} + \mu I_3^{-\frac{1}{3}} dev\mathbf{b}, \tag{35}$$

where $dev(\cdot)$ stands for the deviatoric operator in the current configuration and is given as: $dev(\cdot) = (\cdot) - \frac{1}{3}[\mathbf{g} : (\cdot)]\mathbf{g}^{-1}$.

The loading surfaces are assumed to be given by a von-Mises type of expression in the form:

$$\varphi(\boldsymbol{\tau},\mathbf{g},a,\mathbf{q}) = \sqrt{(\tau^{ij} - q^{ij})(\tau^{kl} - q^{ij})g_{ik}g_{jl} - \tfrac{1}{3}\left(\tau^{kl}g_{kl}\right)^2} - \sqrt{\tfrac{2}{3}}\left(\sigma_y + \delta \bar{H} a\right), \tag{36}$$

where α is a scalar internal variable, that stands for the description of the isotropic hardening of the von-Mises loading surface and $\mathbf{q}$ is a purely deviatoric tensorial internal variable (back-stress), which stands for its kinematic hardening. Finally, σ_y denotes the uniaxial yield stress and δ and $\bar{H}$ are two model parameters related to the hardening properties of the material.

The evolution of the contravariant metric is assumed to be given by a normality flow rule, which resembles the one derived on the basis of the maximum plastic dissipation by Simo [15], within the context of classical multiplicative elastoplasticity:

$$\mu I_3^{-\frac{1}{3}} L_v \mathbf{b} = -2\frac{1}{\beta}\frac{\langle \varphi \rangle}{|\varphi|}\mathbf{n}\langle \mathbf{n} : L_v \mathbf{g} \rangle, \tag{37}$$

where β is an additional parameter. For the rate equations for the evolution of the hardening variables, motivated from the infinitesimal theory (e.g. see Simo and Hughes [38, p. 90]), we consider the following forms:

$$\dot{a} = \sqrt{\frac{2}{3}}\frac{1}{\beta}\frac{\langle \varphi \rangle}{|\varphi|}\langle \mathbf{n} : L_v \mathbf{g} \rangle, \tag{38}$$

$$L_v \mathbf{q} = \frac{2}{3}(1-\delta)\bar{H} L_v \mathbf{b}. \tag{39}$$

Finally, the normal vector $\mathbf{n} = \frac{\partial \varphi}{\partial \mathbf{g}}$ to the loading surfaces, after lengthy computations (see Simo [15]), can be found to be:

$$\mathbf{n} = \frac{\partial \varphi}{\partial \mathbf{g}} = \bar{\mu}\left(\mathbf{m} + \frac{\|\mathrm{dev}\boldsymbol{\tau}\|}{\bar{\mu}}\mathrm{dev}[\mathbf{m}^2]\right), \tag{40}$$

where $\bar{\mu} = \mu I_3^{-\frac{1}{3}} I_1$ and $\mathbf{m} = \frac{\mathrm{dev}\boldsymbol{\tau}}{\|\mathrm{dev}\boldsymbol{\tau}\|}$ is the normal vector to the yield surface in the stress space. An equivalent expression of the model in the reference configuration can be derived in a similar manner like the one discussed in section 2, by a pull-back operation to the basic equations (e.g. [16, 29, 30]).

The proposed model can be implemented numerically by employing a predictor-corrector scheme like the one proposed within the context of a classical elastic-plastic formulation by Simo and Hughes [38, pp. 311-321]. In sharp contrast to the classical elastic-plastic case, *for the model in question the state variables are not constrained to lie within the manifold enclosed by the yield surface, because of the absence of the concept of the yield surface.* Accordingly, *unlike the classical elastic-plastic case where the governing equations define a unilaterally constrained problem of evolution* (e.g. Simo and Ortiz [17], Simo and Hughes [38, pp. 311-321]), in the present case *the governing equations define a differential system which must obey the loading-unloading conditions stated in Section 2* (see Equations (13), (15)). The resulting system is in general highly non-linear and it can be solved by an iterative technique. Algorithmic details related to the crucial role played by the loading-unloading criteria for the numerical implementation of a generalized plasticity model, within the context of the infinitesimal theory, can be found in Panoskaltsis et al. [39]. Additional details, encompassing several algorithmic forms of the time continuous loading-unloading criteria within the context of large deformation plasticity and algorithmic approximations for the Lie derivatives appearing in the formulation, are given in Panoskaltsis et al. [5, 30].

Due to lack of space one problem of large scale plastic flow is considered, that of finite shear. This problem has been used extensively as a testing problem within the context of large deformation constitutive theory (e.g. [4, 5, 18, 40] and is defined by:

$$x^1 = X^1 + \gamma X^2, \;\; x^2 = X^2, \;\; x^3 = X^3,$$

where γ is the shearing parameter. The material parameters are similar to those considered by Simo and Hughes [38, p. 326] (where a related problem, i.e. the elastic- plastic upsetting of an axisymmetric billet is examined):

$$\lambda = 833.33 \text{ MPa}, \;\; \mu = 384{,}62\text{MPa}, \;\; \sigma_y = 1\text{MPa}, \;\; \bar{H} = 3\text{MPa}, \;\; \delta = 0.2.$$

In our example both *isotropic and kinematic hardening mechanisms* are considered. The referential metric is assumed initially to be equal to the Euclidean metric.

The stress-deformation curves predicted by the model for different values of the plastic parameter β are shown in Figures 1 and 2. The predicted response corresponds to stresses which increase monotonically with strain and the oscillating response, reported among others by Atluri [4] in a finite shear problem, due to the use of the (corrotational) Jaumannn rate, does not appear. The salient feature of a generalized plasticity predicted stress-strain curve, according to which reloading, following unloading from a plastic state, results in plastic behavior before the stress level from which the unloading began (see [30, 35]), is verified.

5 Concluding Remarks

A new approach to large deformation plasticity has been proposed. The approach considers the *referential, ''physical'' metric as the primary internal variable*, the time derivative of which accounts for the description of the plastic (dissipative) mechanisms within the material.

The proposed approach has several advantages over the classical approaches to large deformation elastic-plastic theory since:

(1) It is based on a physical quantity namely the physical metric, which can be determined by experimental procedures and does not consider neither the introduction of an “artificial” primary measure accounting for plastic deformation, nor any decomposition of the kinematical quantities in elastic and plastic parts.

(2) By considering the “physical” metric as the primary internal variable and the spatial metric as the control variable, a natural strain-space formulation of the theory is achieved.

(3) It can describe several internal structures, which may differ vastly from the classical Euclidean one.

(4) It constitutes a natural extension of the well established theory of elasticity of Marsden and Hughes [24] to the elastic-plastic range.

(5) It can be extended naturally to a covariant one.

Furthermore, in the course of this development two additional novel features are presented:

(1) The derivation of the Doyle-Ericksen formula within the context of an elastic- plastic continuum.

(2) The derivation of a simple elastic-plastic model within a strain-space formulation and its numerical implementation for the solution of large scale plastic flow problems.

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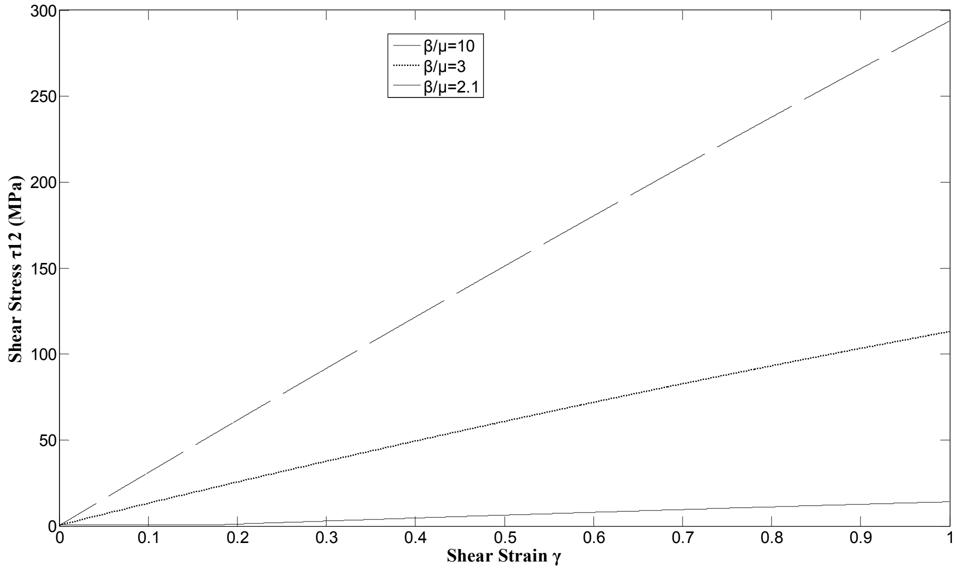

Figure 1: Finite Shear. Shear stress τ_{12} vs. shear strain γ.

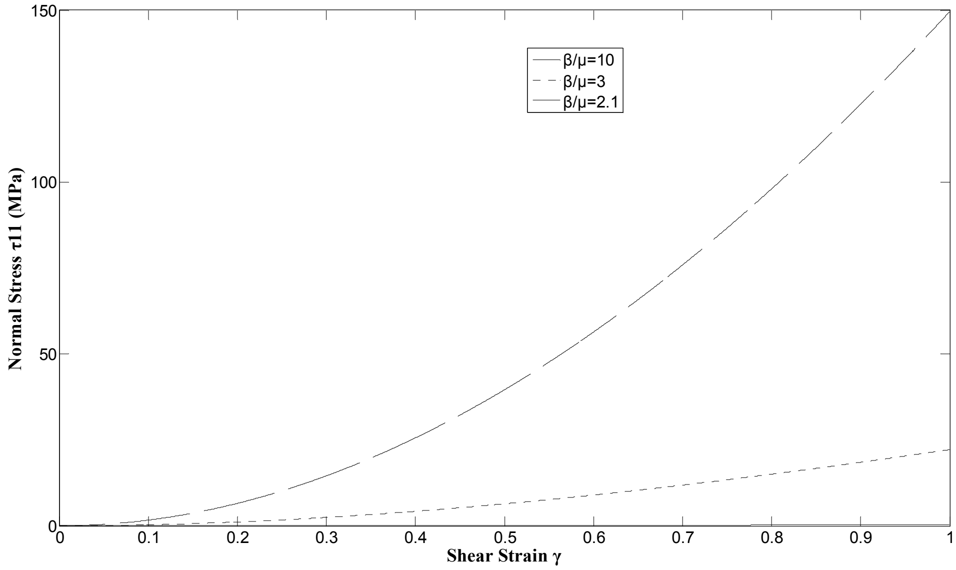

Figure 2: Finite Shear. Normal Stress τ_{11} vs. shear strain γ.